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LETTERS TO THE EDITOR

SIMPLE STABILITY ANALYSES OF A REGENERATIVE CHATTER PROBLEM AND OF A TIME-DELAYED DISPLACEMENT FEEDBACK SDOF SYSTEM

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1. INTRODUCTION

It is well known that the regenerative nature of a cutting force causes machine tool chatter. The regenerative chatter problem can be represented as a time-delayed displacement feedback system, and several techniques developed for control theory can be used to ascertain the stability conditions of the chatter.

Various criteria have been used successfully to determine the stability characteristics of dynamical systems. Most common techniques use the Nyquist criterion to establish whether a certain cutting configuration is stable or unstable [1, 2].

A convenient way to represent the stability condition of a system is to plot its stability chart. A stability chart is a two-dimensional map of stability regions for two feedback parameters, in this case, they are the magnitude of the feedback force and the time-delay.

Using the Nyquist criterion for generating the stability chart is a very time-consuming process. A better technique is to search for the boundaries dividing stable and unstable regions. Tobias [3] had presented several stability charts for machine tool chatter, and used a numerical procedure to generate the chart. In revisiting Tobias's works, it was found that for the particular case of the regenerative chatter, it is possible to solve the problem analytically, and thus to gain better insight into the solutions than given in previous literature. The difficulty in solving the time-delayed feedback problem is that it is not possible to produce an explicit expression for the relationship between the feedback force and the time-delay. However, it is possible to produce two explicit expressions for the relationship between the feedback force and frequency (imaginary part of the eigenvalue), and between the time-delay and frequency. The stability chart can be easily produced from this set of parametric equations. In fact, the relationship between the feedback force and frequency alone provides insights into the shape of the stability regions and also provides explicit expressions for the optima of the stability boundaries which are the delay-independent stability criteria of the system.

This article introduces the use of the frequency as an intermediary term to derive analytically the delay-dependent stability criteria for two types of time-delayed displacement feedback single-degree-of-freedom (s.d.o.f.) systems: the regenerative chatter problem, and a general linear time-delayed displacement feedback system.

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2. REGENERATIVE CHATTER PROBLEMS

2.1. FORMULATION

Numerous stability charts for machine tool chatter have been published since Tobias introduced the regenerative chatter theory [3]. This theory assumed that chatter is associated with a single mode of vibration of the tool and therefore can be modelled as an s.d.o.f system. The cutting force is assumed to be proportional to the chip thickness, so it is dependent on previous cutting conditions. For example, in lathe regenerative chatter, the chip thickness, and hence the cutting force, are dependent on the current position of the tool and on the position of the tool one revolution previously. The fluctuating cutting force, df, is given by

$$\mathrm{df} = -K(x(t) - x(t-T)), \tag{1}$$

where K is a constant called *cutting force factor*, $x(t)$ is the time-dependent displacement of the tool from the equilibrium condition. $x(t-T)$ is the displacement of the tool at time $t-T$, where T is the time delay. In regenerative chatter of a lathe, $T = 2\pi/\Omega$, where Ω is the angular velocity of the work-piece. The cutting force factor K can also be regarded as the gain of a feedback control system. Detailed physical reasoning of this formulation can be found in references [3, 4].

The s.d.o.f. equation of motion of the tool is given as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -K(x(t) - x(t-T)), \tag{2}$$

where m, c, and k are the equivalent mass, damping coefficient and stiffness of the tool.

Equation (2) can be expressed as

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = -Y\omega_n^2(x(t) - x(t-T)), \tag{3}$$

where $\omega_n = (k/m)^{1/2}$ is the natural frequency, $\zeta = c/c_{crit}$ is the damping ratio, $c_{crit} = (4mk)^{1/2}$ is the critical damping coefficient and $Y = K/k$ is the non-dimensional gain factor. The solution of equation (3) has the form $x = A\mathrm{e}^{\lambda t}$, where λ is the complex characteristic value and is a function of ζ. Substituting x into equation (3) gives the following characteristic equation:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 + Y\omega_n^2(1 - \mathrm{e}^{-\lambda T}) = 0. \tag{4}$$

Because of the exponential term in the feedback, finding an analytical solution is difficult, and there is no standard solution to this problem given in any of the textbooks on control theory. Recently, work has been done relating to the delay-independent stability criteria of systems with single time delay [5], and two time delays [6], and of time-delayed dynamic systems of multiple degrees of freedom [7, 8], but an analytical treatment of delay-dependent s.d.o.f. system, as presented in this article, has not been found.

Let $\lambda = \alpha + \mathrm{i}\omega$, where α is the decay rate and ω is the angular frequency. Substituting α and ω into equation (4) and separating the real and imaginary parts gives

$$\alpha^2 - \omega^2 + 2\zeta\omega_n\alpha + \omega_n^2 + Y\omega_n^2(1 - \mathrm{e}^{-\alpha T}\cos\omega T) = 0, \tag{5a}$$

$$2\alpha\omega + 2\zeta\omega_n\omega + Y\omega_n^2\mathrm{e}^{-\alpha T}\sin\omega T = 0. \tag{5b}$$

By introducing the non-dimensional terms $D = \alpha/\omega_n$, $W = \omega/\omega_n$, and $\tau = T\omega_n$, the above equations can be written in non-dimensional form as

$$D^2 - W^2 + 2\zeta D + 1 + Y(1 - \mathrm{e}^{-D\tau}\cos W\tau) = 0, \tag{6a}$$

$$2DW + 2\zeta W + Y\mathrm{e}^{-D\tau}\sin W\tau = 0. \tag{6b}$$

These two simultaneous equations represent the dynamics of the tool, subjected to a cutting force Y, delayed by time τ, and with system damping characteristic ζ. The dynamic characteristics of the response are represented by the decay rate D, and the chatter frequency W. The system is stable if D is negative, and unstable otherwise.

2.2. STABILITY BOUNDARIES

Instead of resorting to checking for stability using various stability criteria for a particular situation as suggested by many numerical procedures, an analytical process is used to determine the onset of instability. This process consists of finding all situations when the decay rate is zero. Thus substituting $D = 0$ into equations (6a) and (6b) gives two simple stability boundary conditions:

$$-W^2 + 1 + Y(1 - \cos W\tau) = 0, \quad 2\zeta W + Y \sin W\tau = 0. \qquad (7a, b)$$

The two parameters associated with the feedback terms are Y and τ, so a complete description of the stability condition of the closed-loop system can be represented on a stability chart by plotting values of Y versus τ that satisfy the two equations (7a) and (7b) for a given damping ratio.

The difficulty in plotting the stability boundary is the presence of the frequency parameter W. In many control problems the characteristic equations are polynomials so W can often be found explicitly. The time-delay problem produces transcendental characteristic equations, hence there is an infinite number of roots that are difficult to express explicitly.

Instead of trying to eliminate W from equations (7a) and (7b), they may be rearranged to produce two equations, explicitly expressing Y in terms of W and ζ, and τ in terms of

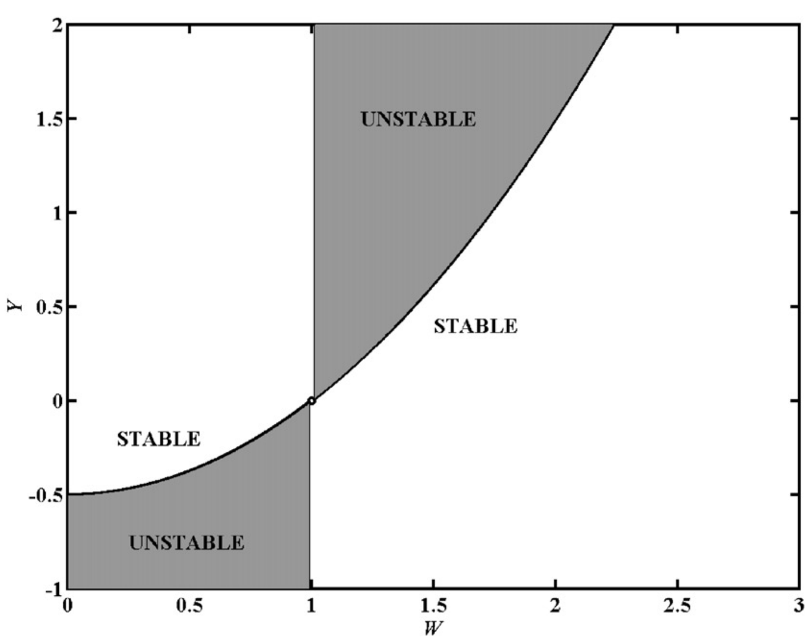

Figure 1. Stability chart for regenerative feedback system on a force Y versus chatter frequency W map, $\zeta = 0$.

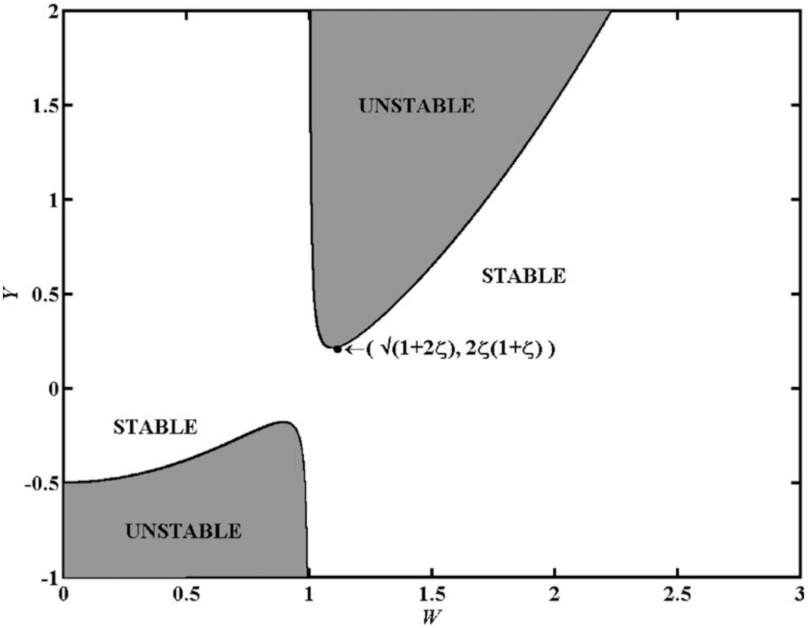

Figure 2. Stability chart for regenerative feedback system on a force Y versus chatter frequency W map, $\zeta = 0.1$.

W and ζ. Rearranging equation (7b) gives

$$Y = \frac{-2\zeta W}{\sin W\tau} \tag{8}$$

and substituting it into equation (7a) gives

$$-W^2 + 1 - 2\zeta W \tan\frac{W\tau}{2} = 0. \tag{9}$$

Hence,

$$\tau = \frac{2}{W}\left(\tan^{-1}\left(\frac{1 - W^2}{2\zeta W}\right) + n\pi\right), \quad n = 0, 1, 2, \ldots, \infty. \tag{10}$$

Substituting equation (10) into equation (8) to eliminate τ, and simplifying to find an explicit expression for Y in terms of W and ζ

$$Y = \frac{-(1 - W^2)^2 - (2\zeta W)^2}{2(1 - W^2)}. \tag{11}$$

The damping ratio ζ may be assumed to be constant for a particular system, and therefore equations (10) and (11) represent a set of parametric equations for the stability boundary in term of τ and Y. Although there are an infinite number relationships between τ and W (equation (10)), there is only one curve for the stability boundary as expressed in equation (11). Valuable insights can be gained from understanding the relationship between Y and W.

For zero damping, $\zeta = 0$, Y is simply a parabola $Y = (W^2 - 1)/2$, with a discontinuity at $W = 1$ (see Figure 1). As damping increases ($\zeta > 0$), the discontinuity at $W = 1$ becomes more pronounced as shown in Figure 2. This curve represents the stability

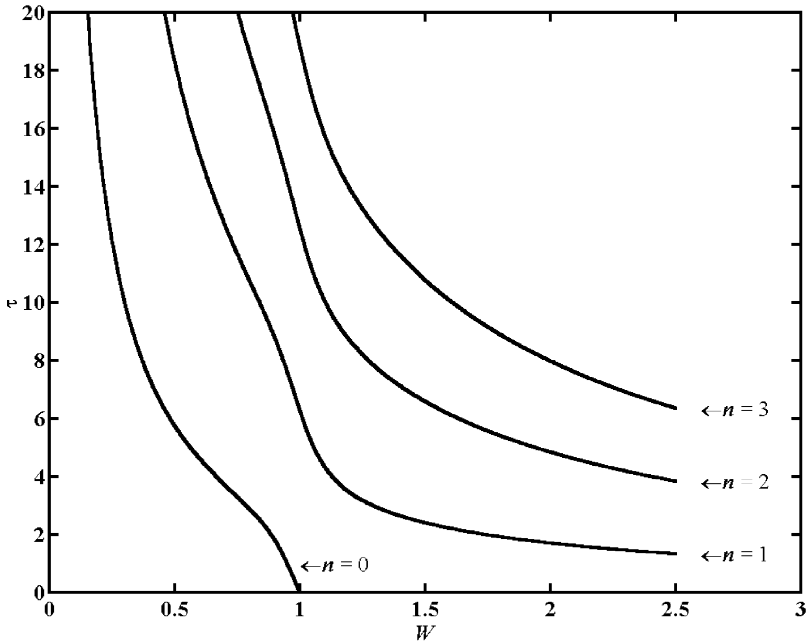

Figure 3. Relationship between time delay τ and chatter frequency W, $\zeta = 0.1$, $n = 0, 1, 2, 3$.

boundary for the feedback gain factor Y as a function of frequency W for a given damping ratio ζ.

Equation (6b) can be used to establish the regions of instability. The closed-loop system is stable if D is negative and unstable otherwise. The second term of equation (6b) indicates that as ζ increases, D becomes more negative. Noting that $e^{-D\tau}$ is always positive, the third term of equation (6b) indicates that if $\sin(W\tau)$ is positive then Y is the same as the damping term—that is the area above the stability boundary line is stable—whereas if $\sin W\tau$ is negative then the area below the stability boundary line is stable. Using equation (10), we have

$$\sin W\tau = \frac{2\zeta W(1 - W^2)}{(1 - W^2)^2 + (2\zeta W)^2}. \tag{12}$$

It is obvious from equation (12) that $\sin(W\tau)$ is positive when $W < 1$ and negative when $W > 1$. Therefore, the region of instability can be established as shown in Figures 1 and 2. For the regenerative chatter problem, Y is positive, so the instability region of interest is the area for $W > 1$.

Another interesting fact that can be deduced from the $Y-W$ relationship is that the minimum point of the instability region can be found explicitly by differentiating equation (11) with respect to W and equating to zero to give

$$\frac{dY}{dW} = \frac{W(W^4 - 2W^2 + 1 - 4\zeta^2)}{(1 - W^2)^2} = 0. \tag{13}$$

There are five roots for this equation: they are $W = 0$, or $W = \pm\sqrt{1 \pm 2\zeta}$. Root $W = \sqrt{1 + 2\zeta}$ relates to the minimum of the instability region of interest, whereas $W = \sqrt{1 - 2\zeta}$ is the maximum of the instability region for $W < 1$. The magnitudes of Y for these two

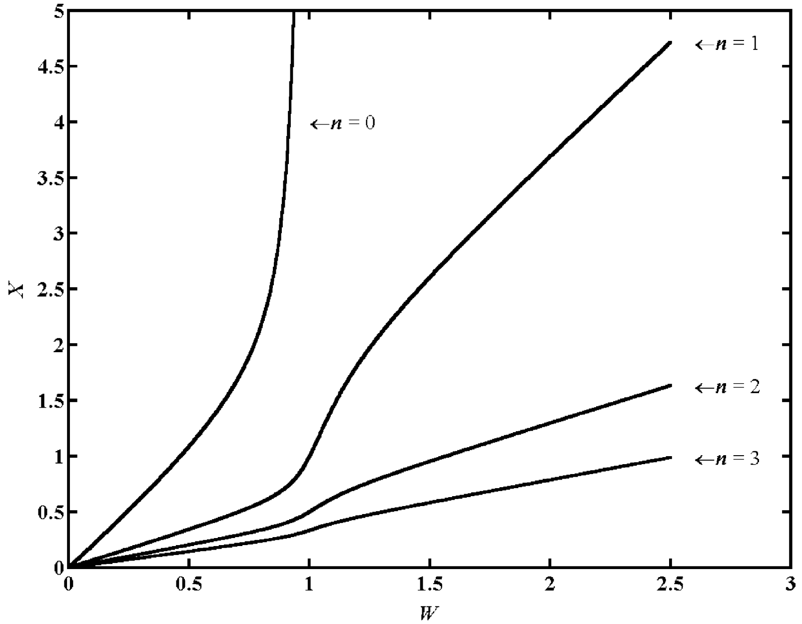

Figure 4. Relationship between repeat frequency X and chatter frequency W, $\zeta = 0.1$, $n = 0$, 1, 2, 3.

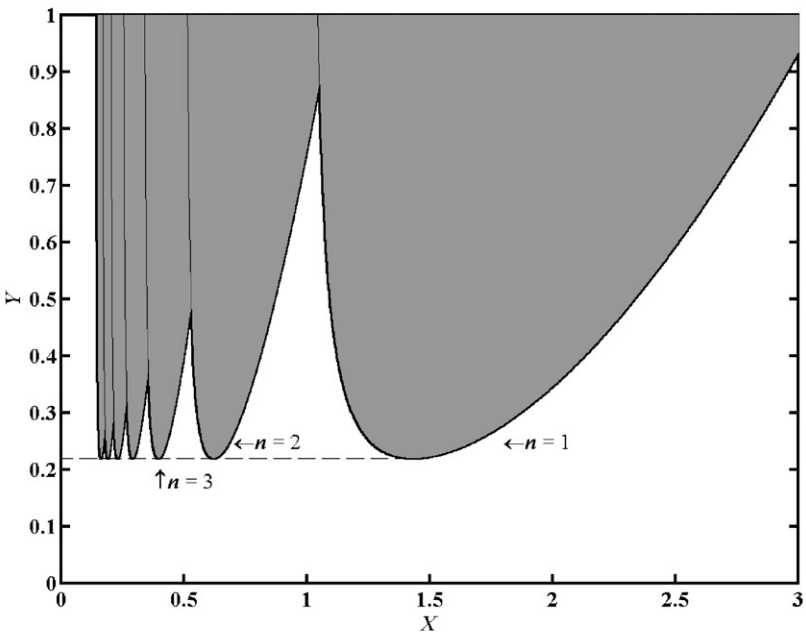

Figure 5. Stability chart for regenerative feedback system on a force Y versus repeat frequency X map, $\zeta = 0.1$, $n = 0-7$.

values of W are

$$Y(\sqrt{1+2\zeta}) = 2\zeta(1+\zeta), \tag{14}$$

$$Y(\sqrt{1-2\zeta}) = -2\zeta(1-\zeta). \tag{15}$$

The relationship between τ and W is complicated by the infinite number of curves produced by the periodic nature of the transcendental equation. Figure 3 shows the curves for $n = 0$, 1, 2 and 3 for $\zeta = 0.1$. In a chatter problem, the time-delay term is usually expressed in term of the repeat frequency of the delayed signal, Ω, or in non-dimensional form $X = \Omega/\omega_n = 2\pi/\tau$. Figure 4 shows the relationship between X and W. The standard stability chart for regenerative chatter can be plotted parametrically using equations (10) and (11) for a constant value of ζ. Figure 5 shows a stability chart for the case $\zeta = 0.1$. This figure shows that the time-delay-independent stability criterion is $Y < 2\zeta(1+\zeta)$. For the case of a very lightly damped system, ζ is close to zero, so the positions of the minimum of the stability regions are close to $\tau = 3/4\pi + 2n\pi$, or $X = 4/(3+4n)$, which are the same as those quoted in much of the literature on chatter (e.g., reference [4]).

3. TIME-DELAYED DISPLACEMENT FEEDBACK

This section uses the same procedure to determine analytically the stability characteristics of an s.d.o.f. time-delayed displacement feedback system. The equation of motion is

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = Y\omega_n^2 x(t-T). \tag{16}$$

Substituting $x = A\mathrm{e}^{\lambda t}$ into equation (16) gives the following characteristic equation:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 - Y\omega_n^2\mathrm{e}^{-\lambda T} = 0. \tag{17}$$

Applying a process similar to that used for the regenerative problem, a set of characteristic equations can be written in non-dimensional form as

$$D^2 - W^2 + 2\zeta D + 1 - Y\mathrm{e}^{-D\tau}\cos W\tau = 0, \tag{18a}$$

$$2DW + 2\zeta W + Y\mathrm{e}^{-D\tau}\sin W\tau = 0. \tag{18b}$$

Substituting $D = 0$ into equations (18a) and (18b) gives two simple stability boundary conditions

$$-W^2 + 1 - Y\cos W\tau = 0, \quad 2\zeta W + Y\sin W\tau = 0. \tag{19a, b}$$

Rearranging equation (19b) gives

$$Y = \frac{-2\zeta W}{\sin W\tau}, \tag{20}$$

and substituting it into equation (19a) gives

$$-W^2 + 1 + \frac{2\zeta W}{\tan W\tau} = 0. \tag{21}$$

Hence,

$$\tau = \frac{1}{W}\left(\tan^{-1}\left(\frac{2\zeta W}{W^2-1}\right) + n\pi\right), \quad n = 0, 1, 2, \ldots, \infty. \tag{22}$$

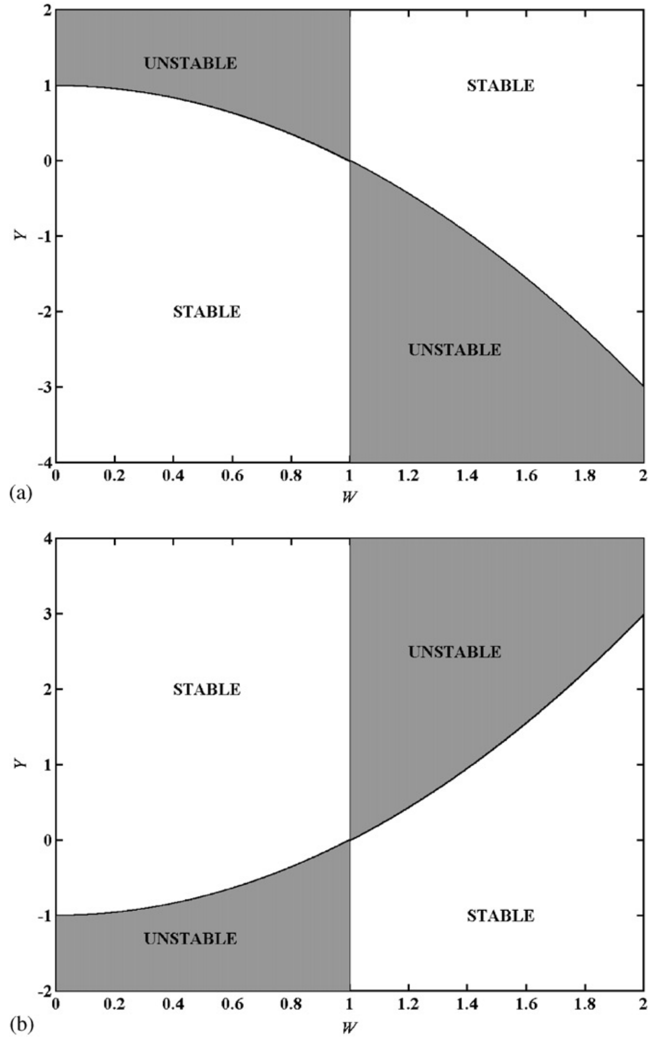

Figure 6. Stability charts for time-delayed feedback system on a force Y versus frequency W map, $\zeta = 0$. (a) For $n = 0, 2, 4, \ldots$; (b) for $n = 1, 3, 5, \ldots$.

This relationship is more complicated than the regenerative chatter case because of the sign changes. The term $\sin W\tau$ can be expressed explicitly as

$$\sin W\tau = \frac{2\zeta W}{\sqrt{(W^2-1)^2 + (2\zeta W)^2}} \quad \text{for} \begin{cases} W < 1, n = 1, 3, 5, \ldots, \\ W > 1, n = 0, 2, 4, \ldots . \end{cases} \tag{23a}$$

$$\sin W\tau = \frac{-2\zeta W}{\sqrt{(W^2-1)^2 + (2\zeta W)^2}} \quad \text{for} \begin{cases} W < 1, n = 0, 2, 4, \ldots, \\ W > 1, n = 1, 3, 5, \ldots . \end{cases} \tag{23b}$$

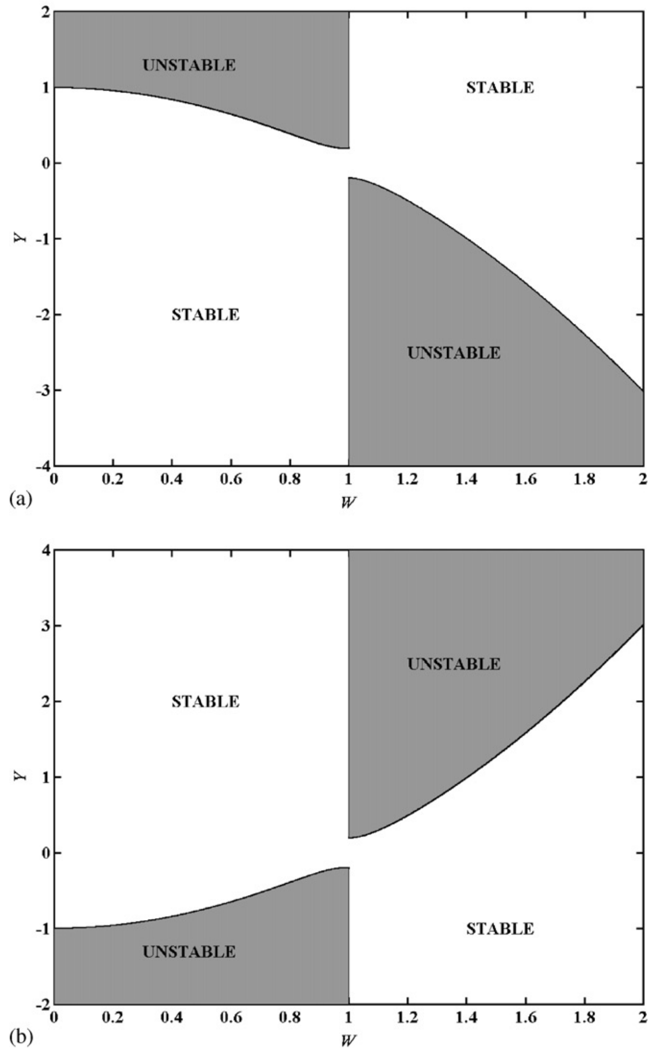

Figure 7. Stability charts for time-delayed feedback system on a force Y versus frequency W map, $\zeta = 0.1$. (a) For $n = 0, 2, 4, \ldots$; (b) for $n = 1, 3, 5, \ldots$.

Substituting equations (23a) and (23b) into equation (20) to eliminate τ, and simplifying, an explicit expression for Y in terms of W and ζ can be found. It is

$$Y = -\sqrt{(W^2 - 1)^2 - (2\zeta W)^2} \quad \text{for} \begin{cases} W < 1, n = 1, 3, 5, \ldots, \\ W > 1, n = 0, 2, 4, \ldots \ . \end{cases} \tag{24a}$$

$$Y = \sqrt{(W^2 - 1)^2 - (2\zeta W)^2} \quad \text{for} \begin{cases} W < 1, n = 0, 2, 4, \ldots, \\ W > 1, n = 1, 3, 5, \ldots \ . \end{cases} \tag{24b}$$

Again, from equation (18b) the stability region is above the boundary line if $\sin W\tau$ is positive, and below the boundary line otherwise. Figures 6(a) and (b) show the stability regions between Y and W for $\zeta = 0$, while Figures 7(a) and (b) show the stability regions for $\zeta = 0.1$.

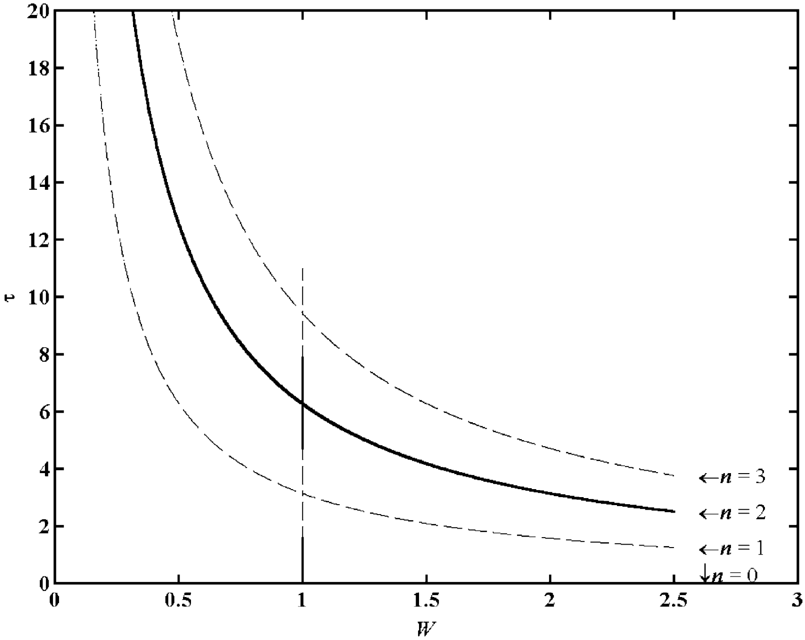

Figure 8. Relationship between time delay τ and frequency W, $\zeta = 0$. —— for $n = 0, 2$; ...... for $n = 1, 3$.

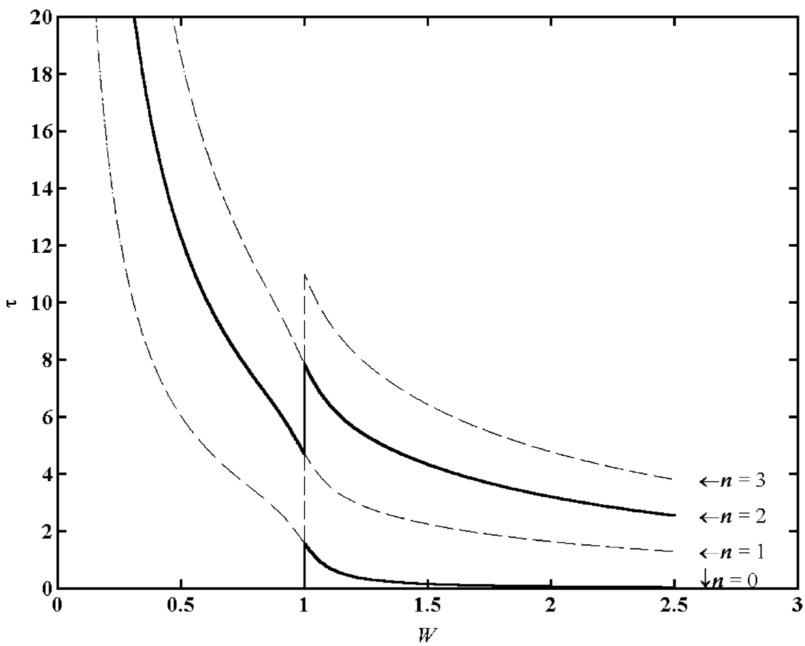

Figure 9. Relationship between time delay τ and frequency W, $\zeta = 0.1$. —— for $n = 0, 2$; ...... for $n = 1, 3$.

Figures 8 and 9 show the relationship between τ and W for $\zeta = 0$ and 0.1, respectively. In Figure 8, as W increases from 0 to 1, τ approaches $n\pi$ but jumps to $(n - \frac{1}{2})\pi$ at $W = 1$.

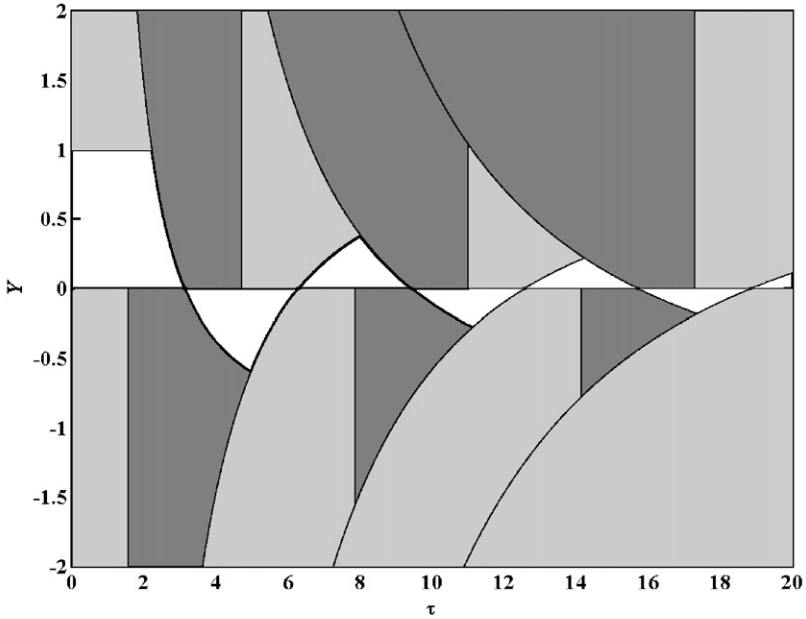

Figure 10. Stability chart for time-delayed feedback system on a force Y versus time delay τ map, $\zeta = 0$. Light areas for $n = 0, 2, 4, 6$ (from left to right); dark areas for $n = 1, 3, 5$ (from left to right).

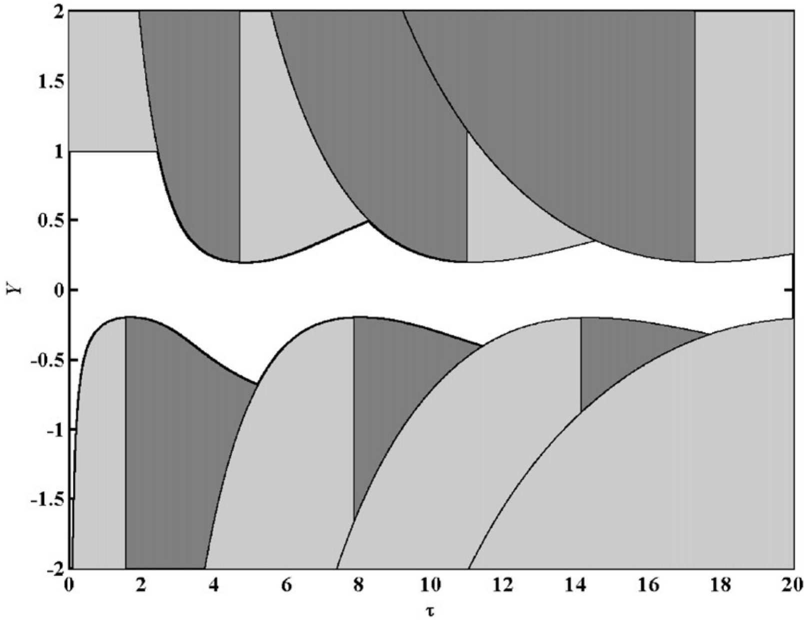

Figure 11. Stability chart for time-delayed feedback system on a force Y versus time delay τ map, $\zeta = 0.1$. Light areas for $n = 0, 2, 4, 6$ (from left to right); dark areas for $n = 1, 3, 5$ (from left to right).

However, as W decreases from ∞ to 1, τ also approaches $n\pi$ but jumps to $(n + 1/2)\pi$ at $W = 1$.

When combined, the stability charts for a delayed displacement feedback system can be produced. Figures 10 and 11 show the relationship between Y and τ for $\zeta = 0$ and 0.1, respectively. Note that the instability regions for $Y < 0$ is for the negative displacement feedback. These charts agree with the numerical simulations.

The delay-independent criteria can be found by differentiating equations (24a) and (24b) with respect to W and equating to zero:

$$\frac{\mathrm{d}Y}{\mathrm{d}W} = 2\frac{W(W^2 - 1 + 2\zeta^2)}{\sqrt{W^4 - 2W^2 + 1 + (2\zeta W)^2}} = 0. \tag{25}$$

So, the minima are at $W = 0$, or $W = \pm\sqrt{1 - 2\zeta^2}$, and the corresponding Y values are

$$Y(\pm\sqrt{1 - 2\zeta^2}) = 2\zeta\sqrt{1 - \zeta^2} \tag{26a}$$

for equation (24a), and

$$Y(\pm\sqrt{1 - 2\zeta^2}) = -2\zeta\sqrt{1 - \zeta^2} \tag{26b}$$

for equation (24b).

Note that the instability area corresponding to $n = 0$ for $Y > 1$ is the static instability due to the steady state error from an overdamped system.

4. CONCLUSION

This article presents a simple procedure to generate stability charts for the regenerative chatter of an s.d.o.f. cutting system, and for a delayed displacement feedback system. This technique solves the closed-loop system equation and finds the conditions that produce zero decay rates, that is the boundaries of stability. The relationship between the feedback force and frequency, and between the time-delay and frequency are found explicitly. The stability chart can be easily produced from this set of parametric equations.

As can be seen from the stability charts that damping reduces the instability regions, therefore it has been one of the key features in machine tool design, and in the design of various feedback systems.

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